Decimal Expansions of Fractions

Sample task from achievethecore.org
Task by Illustrative Mathematics, annotation by Student Achievement Partners

GRADE LEVEL Seven

IN THE STANDARDS 7.NS.A.2d

WHAT WE LIKE ABOUT THIS TASK

Mathematically:

- Focuses on decimal conversions and allows for conjecture about the structure of decimal expansions of fractions
- Leads to student understanding that all rational numbers must either terminate in 0's or eventually repeat
- Uses a variety of unit fractions, allowing students to work from memory, mental math, and/or paper-and-pencil calculations
- Offers opportunities to expand the task to include numerators other than 1 and to compare these to their related unit fractions

In the classroom:

- Requires careful practice with the standard algorithm and attention to precision (MP.6)
- Provides opportunity for students to make conjectures and explore the validity of their conjectures (MP.3)

MAKING THE SHIFTS[1]

Focus	Belongs to the major work[2] of seventh grade
Coherence	Builds on earlier understandings of fractions (5.NF.B.3) and the division algorithm (6.NS.B.2); Lays foundations for future learning about irrational numbers (8.NS.A)
Rigor[3]	Conceptual Understanding: secondary in this task
	Procedural Skill and Fluency: primary in this task
	Application: not addressed in this task

[1] For more information read Shifts for Mathematics.
[2] For more information, see Focus in Grade Seven.
[3] Tasks will often target only one aspect of rigor.

ADDITIONAL THOUGHTS

This task is a great opportunity to illustrate the connection of the decimal expansion of fractions to the base-ten system. For any fraction, $\frac{p}{q}$, that has a finite decimal expansion (i.e., is terminating), there is an equivalent fraction with a denominator that is a power of 10 (i.e., $\frac{p}{q} = \frac{a}{10^n}$). This means that all fractions that result in a terminating decimal have a denominator that is a factor of some power of 10 (10, 100, 1000, ...). For example, $\frac{3}{16}$ is a terminating decimal (0.1875) because 16 is a factor of 10^4 ($16 \times 625 = 10{,}000$). While a formal proof of this fact goes beyond grade 7 expectations, it still serves as a good illustration of how students may look for and make use of structure (MP.7)

For a direct link, go to: http://achievethecore.org/page/1057/decimal-expansion-of-fractions

This task reinforces students' fluency with the division algorithm and understanding of decimals as special fractions, having a denominator that is a power of 10. For more insight on the grade-level concepts addressed in this task, read pages 12 and 13 of the progression document, 6–8, The Number System.

7.NS.2d Decimal Expansions of Fractions

Task

Sarah learned that in order to change a fraction to a decimal, she can use the standard division algorithm and divide the numerator by the denominator. She noticed that for some fractions, like $\frac{1}{4}$ and $\frac{1}{100}$ the algorithm terminates at the hundredths place. For other fractions, like $\frac{1}{8}$, she needed to go to the thousandths place before the remainder disappears. For other fractions, like $\frac{1}{3}$ and $\frac{1}{6}$, the decimal does not terminate. Sarah wonders which fractions have terminating decimals and how she can tell how many decimal places they have.
a. Convert each of the following fractions to decimals to help Sarah look for patterns with her decimal conversions:

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{15}.$$

b. Which fractions on the list have terminating decimals (decimals that eventually end in 0's)? What do the denominators have in common?

c. Which fractions on the list have repeating decimals? What do the denominators have in common?

d. Which fractions $\frac{p}{q}$ (in reduced form) do you think have terminating decimal representations? Which do you think have repeating decimal representations?

IM Commentary

The goal of this task is to convert some fractions to decimals and then make conjectures about which fractions have terminating decimal expansions (as well as the length of those decimals). The teacher may wish to focus more on the conversion process and less on identifying which fractions have terminating or repeating decimal representations. In this case, only part (a) is needed and the key is that all remainders which appear in the long division algorithm are less than the divisor: this means that the decimal must either terminate (when there is a remainder of 0) or repeat (when we see the same non-zero remainder twice). The other parts of this task are looking forward to an additional level of structure, namely identifying which fractions have terminating decimals and which have repeating decimals.

The fractions on this list were deliberately chosen so that many of their decimal expansions will be familiar and will not require extensive calculation. The ones which will require work repeat or terminate quickly. At the teacher's discretion, a calculator could be used to help check work. It is important, however, for students to gain experience with the long division algorithm. Calculators also must be used with care as the only way to know for sure that a decimal is repeating is to perform the long division and see the same sequence of remainders cycling through over and over again.

Students who work on part (d) of this task should be encouraged to investigate their conjectures by testing more examples and, if they are ready, by thinking about the structure of the long division algorithm. Showing that these conjectures are true belongs to the eighth grade standards, 8.NS.1, but students can profitably reflect about this earlier. An explanation for the behavior of decimals examined here can be provided using properties of our base 10 system. A fraction $\frac{a}{b}$, in reduced form, has terminating decimal when $10^n \times \frac{a}{b}$ is a whole number for some positive integer n. This means that $10^n \times a$ is divisible by b. Since a and b share no common factor (other than 1) this means that 10^n must be divisible by b. The only prime factors of 10^n are 2 and 5 so this means that b can only have prime factors of 2 and/or 5. This argument goes beyond the 7th grade standard and is more appropriate for the standard 8.NS.1 but it is important for the teacher to be aware of this structure in case it comes up in discussion: some ideas along these lines are provided in the solution.

Solution

a. We show the long division process on the most difficult of these fractions, namely $\frac{1}{12}$:

$$
\begin{array}{r}
0.0833 \\
12\,\overline{)\,1.0000} \\
0.96 \\
\hline
0.040 \\
0.036 \\
\hline
0.0040 \\
0.0036 \\
\hline
0.0004
\end{array}
$$

Notice that the remainder after subtracting 8×12 (hundredths) is the same as the remainder after subtracting 3×12 (thousandths), namely 4. This means that the 3 in the decimal repeats: we continue to take away 3 groups of 12 (in the ten thousandths place, hundred thousandths place, and so on) and the remainder is always 4. The decimal expansions of all of the fractions are listed below (those which repeat can be found in the same way as $\frac{1}{12}$ above and those which terminate are found when the long division process ends):

$$\begin{aligned}\frac{1}{2} &= 0.5\\ \frac{1}{3} &= 0.\overline{3}\\ \frac{1}{4} &= 0.25\\ \frac{1}{5} &= 0.2\\ \frac{1}{6} &= 0.1\overline{6}\\ \frac{1}{10} &= 0.1\\ \frac{1}{11} &= 0.\overline{09}\\ \frac{1}{12} &= 0.08\overline{3}\\ \frac{1}{15} &= 0.0\overline{6}.\end{aligned}$$

b. The fractions with terminating decimals on the list are:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{10}.$$

The only prime factors of the denominators for each of these fractions are 2 and/or 5.

Taking $\frac{1}{4}$ as an example, we can see where the terminating decimal comes from by observing that 4 is a factor of 100: specifically we use the fact that $4 \times 25 = 100$.

$$\begin{aligned}\frac{1}{4} &= \frac{1 \times 25}{4 \times 25}\\ &= \frac{25}{100}\\ &= 0.25\end{aligned}$$

The last equality comes from the fact that dividing by 100 moves the decimal two places to the left.

c. The fractions with repeating decimals on the list are:

$$\frac{1}{3}, \frac{1}{6}, \frac{1}{11}, \frac{1}{12}, \frac{1}{15}.$$

Each of these fractions has a prime factor different from 2 or 5 in the denominator: 3, 6, 12, and 15 have a prime factor of 3 and 11 has a prime factor of 11. Unlike in the cases in part (b), multiplying by a power of 10 will never result in a whole number here because a factor of 3 or 11 will always remain in the denominator. This means that the decimals do not terminate.

d. The examples studied here indicate that the pattern of a decimal expansion is determined by the denominator (though different numerators should be tried to see if the 1 in the numerator of all of these fractions plays an important role). When the only prime factors of the denominator are 2 and 5 the decimal terminates. When the denominator has a prime factor other than 2 or 5 the decimal eventually repeats. More work would be necessary to see if this always holds: this would mean looking at more fractions with different numerators and denominators and eventually thinking carefully about the division algorithm.